The q-boson realizations of the quantum groups $U_q(D_n)$

1999 J. Phys. A: Math. Gen. 32 6141

(http://iopscience.iop.org/0305-4470/32/34/303)

J. Phys. A: Math. Gen. **32** (1999) 6141–6147. Printed in the UK PII: S0305-4470(99)00688-5

The q-boson realizations of the quantum groups $U_q(D_n)$

Č Burdík†§ and O Navrátil‡

† Department of Mathematics, Czech Technical University, Faculty of Nuclear Sciences and Physical Engineering, Trojanova 13, 120 00 Prague 2, Czech Republic
‡ Department of Mathematics, Faculty of Transportation Sciences, Czech Technical University, Konviktská 20, 110 00 Prague, Czech Republic

E-mail: burdik@dec1.fjfi.cvut.cz

Received 5 January 1999, in final form 6 April 1999

Abstract. We give explicit realization for the quantum enveloping algebras $U_q(D_n)$. To obtain recurrence formulae we simply extend the algebra $U_q(D_n)$ to $U_q(\tilde{D}_n)$, for which $U_q(D_n)$ is a subalgebra. In these formulae the generators of the algebra are expressed by means of $2n-2$ canonical q-boson pairs and one auxiliary representation of $U_q(\tilde{D}_{n-1})$.

1. Introduction

Quantum groups or q-deformed Lie algebras imply some specific deformations of classical Lie algebras. From a mathematical point of view, they are non-commutative associative Hopf algebras. The structure and representation theory of quantum groups were developed extensively by Jimbo [1] and Drinfeld [2]. For a deeper introduction to quantum groups, we would like to recommend some monographs [3, 4].

The q-boson realizations of quantum groups are interesting for both mathematicians and physicists. They are interesting for physicists, since the expression of generators of quantum algebras in terms of elements of the q-oscillator algebra makes it possible to determine physical quantities (for example, Hamiltonians) in terms of the elements of a quantum algebra. Then, using concrete representations of this quantum algebra, we may try to find the spectrum of this physical quantity [5]. Such examples of applications for the cases of classical Lie algebras and their realizations in terms of the usual quantum oscillator are well known in nuclear physics [6, 7] and in solid state physics [8, 9].

The q-boson realizations of quantum groups are interesting for mathematicians, since they can be used for constructing infinite-dimensional representations of quantum algebras. Here it is necessary to note that representations of Lie algebras can be constructed from the corresponding representations of their Lie groups and the latter representations are simply constructed by the method of induced representations when using representations of the corresponding subgroups. In the case of quantum algebras, no method exists for the construction of infinite-dimensional irreducible representations of a quantum algebra from representations of the corresponding quantum group. This fact makes the results of this paper very important from the point of view of the theory of infinite-dimensional representations of quantum algebras.

§ Author to whom correspondence should be addressed.

0305-4470/99/346141+07$30.00

Following the pioneering works [10, 11], q-boson realizations of the quantum groups were constructed in many papers [12–19]. In our papers [20–22] we studied the realizations of $U_q(sl(2))$, $U_q(gl(n))$, and $U_q(B_n)$. Some special Dyson-type realizations were studied in [23].

2. Preliminaries

In this paper, we use the definition of a quantum group [1] given by relations among its Chevalley generators.

Let $\mathcal{L}$ be a simple finite-dimensional Lie algebra. $\boldsymbol{A} = (a_{ij})$ is its Cartan matrix. Let q be an independent variable, $\mathcal{A} = C[q, q^{-1}]$ and $\mathcal{C}(q)$ is a division field of $\mathcal{A}$. For $n \in N$ and $d \in N$ we denote

$$[n]_d = \frac{q^{nd} - q^{-nd}}{q^d - q^{-d}} \in \mathcal{A} \tag{1}$$

$$[n]_d! = [n]_d \cdot [n-1]_d \cdot \ldots \cdot [1]_d \tag{2}$$

and

$$\begin{bmatrix} n \\ j \end{bmatrix}_d = \frac{[n]_d!}{[n-j]_d! \cdot [j]_d!}. \tag{3}$$

If $d = 1$ we omit the subscript d.

Let d_i be the smallest natural numbers such that matrix $(d_i a_{ij})$ is symmetric and positive.

The quantized universal enveloping algebra $U_q(\mathcal{L})$ of a semisimple Lie algebra $\mathcal{L}$ on the field $\mathcal{C}(q)$ is defined by Chevalley generators E_i, F_i, K_i and K_i^{-1}, $i = 1, \ldots, n$, which satisfy the commutation relations

$$\begin{aligned}
&K_i K_j = K_j K_i \qquad K_i K_i^{-1} = K_i^{-1} K_i = 1 \\
&K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \qquad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j \\
&E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \\
&\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \qquad i \neq j \\
&\sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \qquad i \neq j
\end{aligned} \tag{4}$$

where $q_i = q^{d_i}$.

In the case of $U_q(D_n)$ the Cartan matrix is

$$\boldsymbol{A} = (a_{ij}) = \begin{pmatrix}
2 & -1 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots \\
-1 & 2 & -1 & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & -1 & 2 & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & 2 & -1 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & -1 & 2 & -1 & -1 \\
\ldots & \ldots & \ldots & \ldots & 0 & -1 & 2 & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & -1 & 0 & 2
\end{pmatrix}.$$

In this case we have $d_i = 1$ for $i = 1, \ldots, n$, which gives $q_i = q$ for $i = 1, \ldots, n$.

The commutation relations for Chevalley generators are

$$\begin{aligned}
&K_i E_j K_i^{-1} = q^{a_{ij}} E_j \qquad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j \qquad i = 1, \ldots, n \\
&E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q - q^{-1}) \qquad i = 1, \ldots, n \\
&E_i^2 E_{i\pm1} - (q + q^{-1}) E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0 \qquad i = 1, \ldots, n-1 \\
&E_n^2 E_{n-2} - (q + q^{-1}) E_n E_{n-2} E_n - E_{n-2} E_n^2 = 0 \\
&E_{n-2}^2 E_n - (q + q^{-1}) E_{n-2} E_n E_{n-2} - E_n E_{n-2}^2 = 0
\end{aligned} \tag{5}$$

and similar commutation relations for F_i.

In paper [24] we showed the construction of boson realizations for the simple Lie algebra $\mathcal{L}$, by using the induced representations. We also used this method for quantum algebra [21,25]. The difference is that for the quantum algebra case we do not use the $\mathcal{W}$ algebra but its q-deformed version $\mathcal{H}$ [10, 11].

The algebra $\mathcal{H}$, is the associative algebra over field $\mathcal{C}(q)$ which is generated by elements of a^+, $a^- = a$, q^x and q^{-x}, which satisfy the commutation relations

$$\begin{aligned}
&aa^+ - q^{-1}a^+a = q^x \qquad a^+ - qaa^+a = q^{-x} \\
&q^x a^+ q^{-x} = qa^+ \qquad q^x a q^{-x} = q^{-1}a \\
&q^x q^{-x} = q^{-x} q^x = 1.
\end{aligned} \tag{6}$$

This algebra has faithful representation on a vector space with basic elements $\{|n\rangle$, where $n = 0, 1, \ldots\}$:

$$\begin{aligned}
&q^x|n\rangle = q^n|n\rangle \\
&a^+|n\rangle = |n+1\rangle \\
&a|n\rangle = [n]|n-1\rangle.
\end{aligned} \tag{7}$$

Definition. *Let $U_q(\mathcal{L})$ be a quantum algebra and $U_q(\mathcal{L}_0)$ its subalgebra. A realization of the quantum algebra $U_q(\mathcal{L})$ is a homomorphism*

$$\tau : U_q(\mathcal{L}) \longrightarrow \mathcal{H}^n \otimes U_q(\mathcal{L}_0)$$

where $\mathcal{H}^n$ is an n-fold tensor product of the Hayashi algebras.

3. Construction of the realization of $U_q(D_n)$

If we take the new generator K_0 with commutation relations

$$K_0 E_1 K_0^{-1} = q^{-1} E_1 \qquad K_0 F_1 K_0^{-1} = q F_1$$

and the other commutation relations are zero, we obtain the extension of the algebra $U_q(D_n)$, which we denote as $U_q(\tilde{D}_n)$. Evidently, $U_q(D_n)$ is the subalgebra $U_q(\tilde{D}_n)$ and $U_q(\tilde{D}_{n-1}) \subset U_q(\tilde{D}_n)$ is a subalgebra.

Let $U_q(\tilde{D}_{n-1})$ be a subalgebra of $U_q(\tilde{D}_n)$, which is generated by elements K_1, E_j, F_j, K_j and K_j^{-1}, $j = 2, \ldots, n$, and φ be an arbitrary representation of $U_q(\tilde{D}_{n-1})$.

Let $U_q(\mathcal{L}_0)$ be the extension of the algebra $U_q(\tilde{D}_{n-1})$ by element F_1. The representation φ can be extended to the representation of $U_q(\mathcal{L}_0)$ if we put $\varphi(F_1) = 0$.

We denote

$$\begin{aligned}
&X_1 = E_1 \\
&X_k = E_k X_{k-1} - q^{-1} X_{k-1} E_k \qquad k = 2, \ldots, n-1 \\
&Y_{n-1} = E_n X_{n-2} - q^{-1} X_{n-2} E_n \\
&Y_k = E_{k+1} Y_{k+1} - q^{-1} Y_{k+1} E_{k+1} \qquad k = 1, \ldots, n-2
\end{aligned}$$

and further

$$|x, y\rangle = |x_1, x_2, \ldots, x_{n-1}, y_{n-1}, \ldots, y_1\rangle = X_1^{x_1} \cdot X_2^{x_2} \ldots X_{n-1}^{x_{n-1}} \cdot Y_{n-1}^{y_{n-1}} \cdot \ldots Y_1^{y_1}.$$

In the algebra $U_q(\tilde{D}_{n-1})$ we specify the elements

$$\begin{aligned}
&G_2 = E_2\\
&G_k = E_k G_{k-1} - q^{-1} G_{k-1} E_k \qquad k = 3, \ldots, n-1\\
&H_{n-1} = E_n G_{n-2} - q^{-1} G_{n-2} E_n\\
&H_k = E_{k+1} H_{k+1} - q^{-1} H_{k+1} E_{k+1} \qquad k = 2, \ldots, n-2.
\end{aligned}$$

As we see, these elements appear in our induced representation, see theorem 1. In the algebra $U_q(\tilde{D}_{n-1})$ they play the same role as X_i and Y_j in the algebra $U_q(\tilde{D}_n)$.

Starting from now we take the case $n > 3$.

The induced representation ρ, of quantum algebra $U_q(\tilde{D}_n)$ is generated on the space with the basis

$$|x, y; v\rangle = |x, y\rangle \otimes v$$

where the $\{v \in V\}$ form a basis of representation φ of the subalgebra $U_q(D_{n-1})$, which is extended to the representation of $U_q(\mathcal{L}_0)$, as above.

By using the commutation relations (5) and mathematical induction we now obtain the following lemma.

Lemma. *For any $k = 0, 1, \ldots$ and $r = 2, \ldots, n-1$ the following formulae hold:*

$$\begin{aligned}
&E_r X_{r-1}^k = q^{-k} X_{r-1}^k E_r + [k] X_{r-1}^{k-1} X_r\\
&E_r Y_r^k = q^{-k} Y_r^k E_r + [k] Y_r^{k-1} Y_{r-1}\\
&E_n X_{n-2}^k = q^{-k} X_{n-2}^k E_n + [k] X_{n-2}^{k-1} Y_{n-1}\\
&E_n X_{n-1}^k = q^{-k} X_{n-1}^k E_n + [k] X_{n-1}^{k-1} Y_{n-2}\\
&F_n Y_{n-1}^k = Y_{n-1}^k F_n + [k] X_{n-2} Y_{n-1}^{k-1} K_n^{-1}\\
&F_n Y_{n-2}^k = Y_{n-2}^k F_n + [k] X_{n-1} Y_{n-2}^{k-1} K_n^{-1}\\
&F_r X_r^k = X_r^k F_r + [k] X_{r-1} X_r^{k-1} K_r^{-1}\\
&F_r Y_{r-1}^k = Y_{r-1}^k F_r + [k] Y_r Y_{r-1}^{k-1} K_r^{-1}\\
&F_1 X_1^k = X_1 F_1^k - (q - q^{-1})^{-1} [k] X_1^{k-1} (q^{k-1} K_1 - q^{-k+1} K_1^{-1})\\
&F_1 X_r^k = X_r^k F_1 - q^{k-2} [k] X_r^{k-1} G_r K_1\\
&F_1 Y_r^k = Y_r^k F_1 - q^{k-2} [k] Y_r^{k-1} H_r K_1\\
&F_1 Y_1^k = Y_1^k F_1 - q^{k-1} [k] Y_1^{k-1} K_1 (q G_2 H_2 - q^{-1} H_2 G_2)\\
&G_r Y_r^k = q^{-k} Y_r^k G_r + (-1)^r q^{-r} [k] Y_r^{k-1} \Omega_{r-1}\\
&\Omega_r Y_r^k = (-1)^{r+1} (q - q^{-1}) q^{r-k} Y_r^{k+1} G_r + q^{-k} Y_r^k \Omega_{r-1}
\end{aligned}$$

where $\Omega_r = q^2 Y_1 - (q - q^{-1}) \sum_{s=2}^{r} (-1)^s q^s Y_s G_s$.

We omit the details of the calculations and write the result for the action of induced representation ρ on the basis.

Theorem 1. *Let $n > 3$ and $r = 2, \ldots, n-1$, then the formulae*

$$E_1 |x, y\rangle \otimes v = |x + 1_1, y\rangle \otimes v$$

$$E_r |x, y\rangle \otimes v = [x_{r-1}] |x - 1_{r-1} + 1_r, y\rangle \otimes v + q^{x_r - x_{r-1}} [y_r] |x, y - 1_r + 1_{r-1}\rangle \otimes v$$

$$
\begin{aligned}
&+q^{x_r-x_{r-1}-y_r+y_{r-1}}|x, y\rangle \otimes \varphi(E_r)v \\
E_n|x, y\rangle \otimes v &= [x_{n-2}]|x-1_{n-2}, y+1_{n-1}\rangle \otimes v + q^{y_{n-1}-x_{n-2}}[x_{n-1}]|x-1_{n-1}, y+1_{n-2}\rangle \otimes v \\
&+q^{y_{n-1}+y_{n-2}-x_{n-1}-x_{n-2}}|x, y\rangle \otimes \varphi(E_n)v \\
K_1|x, y\rangle \otimes v &= q^{x_1+X_1^{n-1}+Y_2^{n-1}}|x, y\rangle \otimes \varphi(K_1)v \\
K_r|x, y\rangle \otimes v &= q^{x_r-x_{r-1}+y_{r-1}-y_r}|x, y\rangle \otimes \varphi(K_r)v \\
K_n|x, y\rangle \otimes v &= q^{y_{n-2}+y_{n-1}-x_{n-2}-x_{n-1}}|x, y\rangle \otimes \varphi(K_n)v \\
K_0|x, y\rangle \otimes v &= q^{\lambda_0-X_1^{n-1}-Y_1^{n-1}}|x, y\rangle \otimes v \\
F_n|x, y\rangle \otimes v &= q^{x_{n-1}-y_{n-2}}[y_{n-1}]|x+1_{n-2}, y-1_{n-1}\rangle \otimes \varphi(K_n^{-1})v \\
&+[y_{n-2}]|x+1_{n-1}, y-1_{n-2}\rangle \otimes \varphi(K_n^{-1})v + |x, y\rangle \otimes \varphi(F_n)v \\
F_r|x, y\rangle \otimes v &= q^{y_r-y_{r-1}}[x_r]|x-1_r+1_{r-1}, y\rangle \otimes \varphi(K_r^{-1})v \\
&+[y_{r-1}]|x, y+1_r-1_{r-1}\rangle \otimes \varphi(K_r^{-1})v + |x, y\rangle \otimes \varphi(F_r)v \\
F_1|x, y\rangle \otimes v &= -(q-q^{-1})^{-1}[x_1]|x-1_1, y\rangle \otimes [q^{X_1^{n-1}+Y_2^{n-1}-1}\varphi(K_1) \\
&-q^{-X_1^{n-1}-Y_2^{n-1}+1}\varphi(K_1^{-1})]v - \sum_{r=2}^{n-1} q^{Y_1^r-2}[y_r]|x, y-1_r\rangle \otimes \varphi(H_rK_1)v \\
&-q^{y_1-1}[y_1]|x, y-1_1\rangle \otimes \varphi[K_1(qG_2H_2-q^{-1}H_2G_2)]v \\
&-\sum_{r=2}^{n-1} q^{X_r^{n-1}+Y_1^{n-1}-y_r-2}[x_r]|x-1_r, y\rangle \otimes \varphi(G_rK_1)v \\
&-\sum_{r=2}^{n-1}(-1)^r q^{X_r^{n-1}+Y_r^{n-1}-r}[x_r][y_r] \times |x-1_r, y-1_r+1_1\rangle \otimes \varphi(K_1)v \\
&+(q-q^{-1})\sum_{r=3}^{n-1}\sum_{s=2}^{r-1}(-1)^{r+s}q^{X_r^{n-1}+Y_r^{n-1}+Y_1^{s-1}+s-r-2} \\
&\times[x_r][y_r]|x-1_r, y-1_r+1_s\rangle \otimes \varphi(G_sK_1)v
\end{aligned}
$$

where

$$
X_r^s = \sum_{k=r}^{s} x_k \qquad \text{and} \qquad Y_r^s = \sum_{k=r}^{s} y_k
$$

give the induced representation of the algebra $U_q(\tilde{D}_n)$.

From the explicit form of the induced representation we can see that it is possible to rewrite it by using representation (7) of the Hayashi generators a_i^+, a_i, q^{x_i} and q^{-x_i}, b_i^+, b_i, q^{y_i} and q^{-y_i}, where $i, j = 1, \ldots, n-1$, and representation φ of the generators $U_q(\tilde{D}_{n-1})$. The facts that representation (7) is faithful and φ can be arbitrary, give us the following theorem.

Theorem 2. *Let $n > 3$ and $r = 2, \ldots, n-1$, then the mapping $\tau : U_q(\tilde{D}_n) \to \mathcal{H}^{2n-1} \otimes U_q(\tilde{D}_{n-1})$ defined by formulae:*

$$
\begin{aligned}
\rho(E_1) &= a_1^+ \\
\rho(E_r) &= a_{r-1}a_r^+ + q^{x_r-x_{r-1}}b_rb_{r-1}^+ + q^{x_r-x_{r-1}-y_r+y_{r-1}}\varphi(E_r) \\
\rho(E_n) &= a_{n-2}b_{n-1}^+ + q^{y_{n-1}-x_{n-2}}a_{n-1}b_{n-2}^+ + q^{y_{n-1}+y_{n-2}-x_{n-1}-x_{n-2}}\varphi(E_n) \\
\rho(K_1) &= q^{x_1+X_1^{n-1}+Y_2^{n-1}}\varphi(K_1) \\
\rho(K_r) &= q^{x_r-x_{r-1}-y_r+y_{r-1}}\varphi(K_r) \\
\rho(K_n) &= q^{y_{n-1}+y_{n-2}-x_{n-1}-x_{n-2}}\varphi(K_n)
\end{aligned}
$$

$$\rho(K_0) = q^{\lambda_0 - X_1^{n-1} - Y_1^{n-1}}$$
$$\rho(F_n) = q^{x_{n-1}-y_{n-2}} a_{n-2}^+ b_{n-1}\varphi(K_n^{-1}) + a_{n-1}^+ b_{n-2}\varphi(K_n^{-1}) + \varphi(F_n)$$
$$\rho(F_r) = q^{y_r - y_{r-1}} a_r a_{r-1}^+ \varphi(K_r^{-1}) + b_r^+ b_{r-1}\varphi(K_r^{-1}) + \varphi(F_r)$$
$$\begin{aligned}\rho(F_1) = -(q-q^{-1})^{-1}[q^{X_1^{n-1}+Y_2^{n-1}}\varphi(K_1) - q^{-X_1^{n-1}-Y_2^{n-1}}\varphi(K_1^{-1})]a_1 \\ -\sum_{r=2}^{n-1} q^{Y_1^r} b_r \varphi(K_1 H_r) - q^{y_1} b_1 \varphi[K_1(qG_2H_2 - q^{-1}H_2G_2)] \\ -\sum_{r=2}^{n-1} q^{X_r^{n-1}+Y_r^{n-1}-y_r} a_r \varphi(K_1 G_r) - \sum_{r=2}^{n-1}(-1)^r q^{X_r^{n-1}+Y_r^{n-1}+2-r} a_r b_r b_1^+ \varphi(K_1) \\ +(q-q^{-1})\sum_{r=3}^{n-1}\sum_{s=2}^{r-1}(-1)^{r+s} q^{X_r^{n-1}+Y_r^{n-1}+Y_1^{s-1}+s-r+1} a_r b_r b_s^+ \varphi(K_1 G_s)\end{aligned}$$

is a realization of the quantum group $U_q(\tilde{D}_n)$. In these formulae we mean $q^{X_r^s} = \prod_{i=r}^s q^{x_i}$ and $q^{Y_r^s} = \prod_{i=r}^s q^{y_i}$.

Because $U_q(D_3)$ is isomorphic to $U_q(A_3)$, we can take the realization from our previous paper [7]:

$$\rho(E_1) = a_1 a_2^+ + q^{x_2 - x_1}\varphi(E_1)$$
$$\rho(E_2) = a_1^+$$
$$\rho(E_3) = a_2 a_3^+ + q^{x_3 - x_1}\varphi(E_3)$$
$$\rho(K_1) = q^{x_2 - x_1}\varphi(K_1)$$
$$\rho(K_2) = q^{x_1 + X_1^3}\varphi(K_2)$$
$$\rho(K_3) = q^{x_3 - x_2}\varphi(K_3)$$
$$\rho(F_3) = a_3 a_2^+ \varphi(K_3^{-1}) + \varphi(F_3)$$
$$\rho(F_1) = a_2 a_1^+ \varphi(K_1^{-1}) + \varphi(F_1)$$
$$\rho(F_2) = -(q-q^{-1})^{-1}[q^{X_1^3}\varphi(K_2) - q^{-X_1^3}\varphi(K_2^{-1}]a_1 - q^{x_2+x_3} a_a \varphi(K_2G_2) - q^{x_3} a_3 \varphi(K_2 G_3)$$
$$\rho(K_0) = q^{\lambda_0 - x_2 - x_3}$$

where $G_2 = E_1$ and $G_3 = E_3E_1 - q^{-1}E_1E_3$ is the realization of the quantum group $U_q(A_2)$ generated by E_1, E_3, F_1, F_3, K_1 and K_3 extended by generator K_2.

Acknowledgment

Partial support from the grant 202/96/0218 provided by the Czech Grant Agency is gratefully acknowledged.

References

[1] Jimbo M 1986 *Lett. Math. Phys.* **10** 63
Jimbo M 1986 *Lett. Math. Phys.* **11** 247
[2] Drinfeld V 1986 *Proc. Int. Congress of Mathematicians (Berkeley)* (New York: Academic) p 798
[3] Chari V and Pressley A 1994 *Guide to Quantum Groups* (Cambridge: Cambridge University Press)
[4] Majid S 1995 *Foundation of Quantum Groups* (Cambridge: Cambridge University Press)
[5] Del Sol Mesa Ă, Loyola G, Moshinsky M and Velázquez V 1993 *J. Phys. A: Math. Gen.* **26** 1147
[6] Rink P and Shuck P 1980 *The Nuclear Many-Body Problem* (New York: Springer)

[7] Biedenharn L C and Louk J D 1981 Angular momentum in quantum physics *Encyclopedia of Mathematics and its Application* 8 (Reading, MA: Addison-Wesley)
[8] Gaspers W J 1989 *Spin Systems* (Singapore: World Scientific)
[9] Kittel C 1963 *Quantum Theory of Solids* (New York: Wiley)
[10] Macfarlane A J 1989 *J. Phys. A: Math. Gen.* **22** 4581
[11] Biedenharn L C 1989 *J. Phys. A: Math. Gen.* **22** L873
[12] Chaichian M, Ellinas D and Kulish P P 1990 *Phys. Rev. Lett.* **65** 980
[13] Quesne C 1991 *Phys. Lett.* A **153** 303
[14] Chakrabarti R and Jagannathan R 1991 *J. Phys. A: Math. Gen.* **24** L711
[15] Katriel J and Solomon A I 1991 *J. Phys. A: Math. Gen.* **24** 2093
[16] Yu Z R 1991 *J. Phys. A: Math. Gen.* **24** L1321
[17] Kundu A and Mallik B 1991 *Phys. Lett.* A **156** 175
[18] Pan F 1991 *Chin. Phys. Lett.* **8** 56
[19] da-Providencia J 1993 *J. Phys. A: Math. Gen.* **26** 5845
[20] Burdík Č and Navrátil O 1990 *J. Phys. A: Math. Gen.* **23** L1205
[21] Burdík Č, Černý L and Navrátil O 1993 *J. Phys. A: Math. Gen.* **26** L83
[22] Burdík Č and Navrátil O 1998 *Czech. J. Phys.* B **47** 1301
[23] Palev T D 1998 *J. Phys. A: Math. Gen.* **31** 5145
[24] Burdík Č 1985 *J. Phys. A: Math. Gen.* **18** 3101
[25] Hang-Chen Fu and Mo-Lin Ge 1992 *J. Math. Phys.* **33** 427